Math 4200
Monday November 23
Chapter 5: Conformal maps. This is a more in-depth return to ideas we began the course with. There's probably more than one day's worth of material in today's notes. We won't rush.

Announcements:

Recall the chain rule for curves:

If $f$ is analytic at $z_0$ and
$\gamma : I \to \mathbb{C}$ a differentiable path, with $\gamma(t_0) = z_0$,

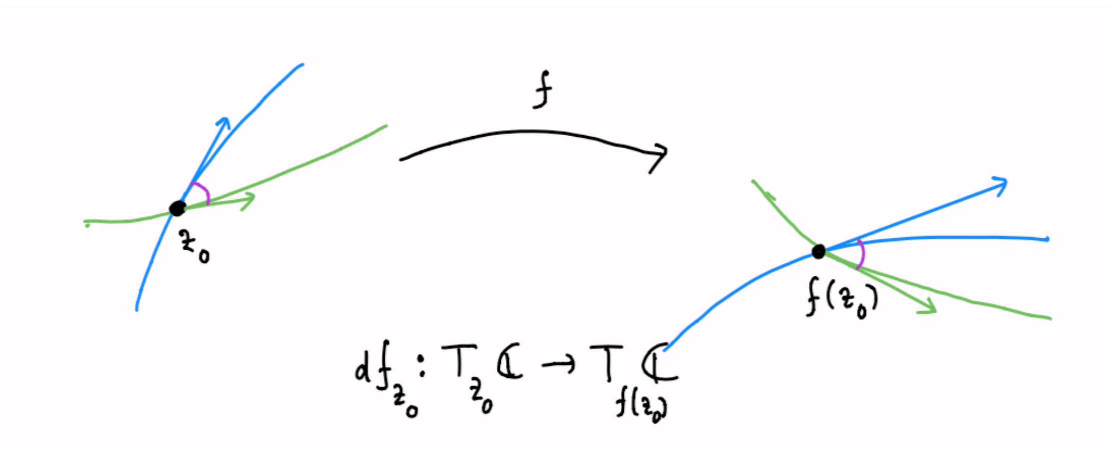
Then

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0) = f'(z_0)\gamma'(t_0).$$

In other words, the differential map $df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}$

$$df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}$$
$$\gamma'(t_0) \to f'(z_0)\gamma'(t_0)$$

converts tangent vectors based at $z_0$ into ones based at $f(z_0)$, via a rotation amount $\arg f'(z_0)$ and a scaling by $|f'(z_0)|$.

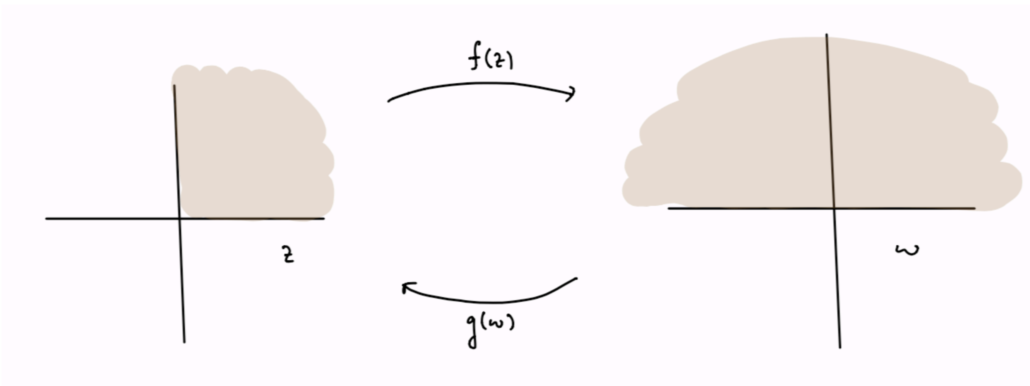



We called a differentiable map $f : A \to \mathbb{C}$ *conformal* on $A$ if this rotation and scaling property holds $\forall\, z \in A$. And we know by the Cauchy-Riemann equations that this rotation-dilation property on tangent vectors is equivalent to $f : A \to \mathbb{C}$ analytic, with $f'(z) \neq 0 \;\forall\, z \in A$.

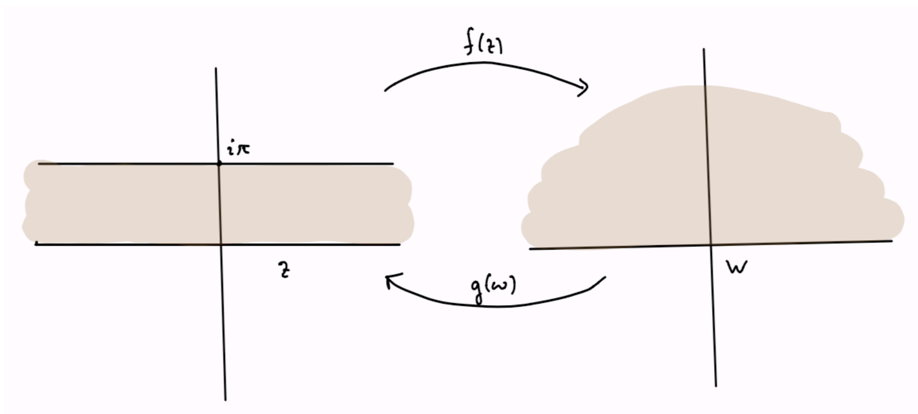
In Chapter 5 we are interested in finding bijective conformal maps between open connected domains $A, B \subseteq \mathbb{C}$. In such cases we will call $A$ and $B$ *conformally equivalent.* Applications include partial differential equations and geometry.

*Example 1* How many conformal bijections can we find between the $1^{st}$ quadrant and the upper half plane? Just to be clear, we're focusing on the open domains and not worrying about boundary behavior at this point.

$f(z) = z^2$, $g(w) = \sqrt{w}$ is one pair. What others can we find?

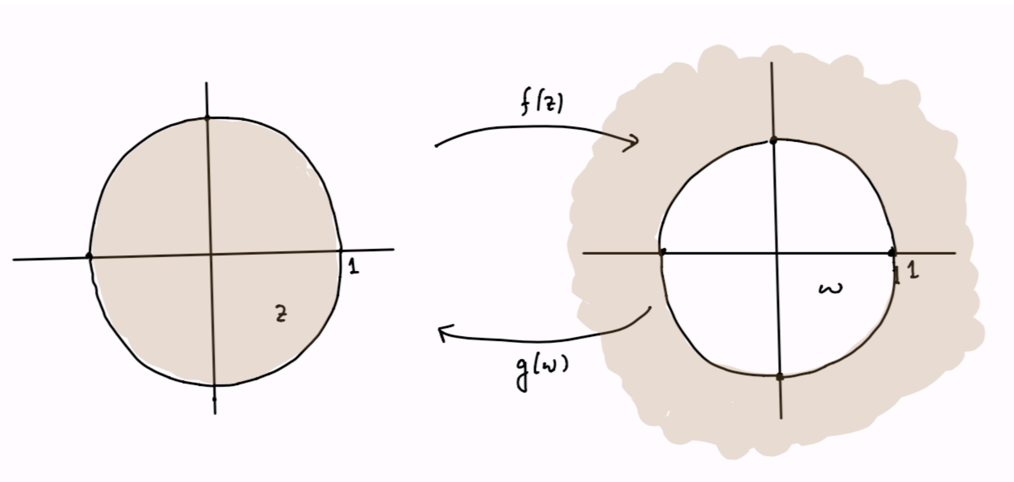


*Example 2* Same question for $A$ equal the strip for which $0 < \operatorname{Im}(z) < \pi$, and $B$ equal to the upper half plane.
one pair: $f(z) = e^z$, $g(w) = \log(w)$.

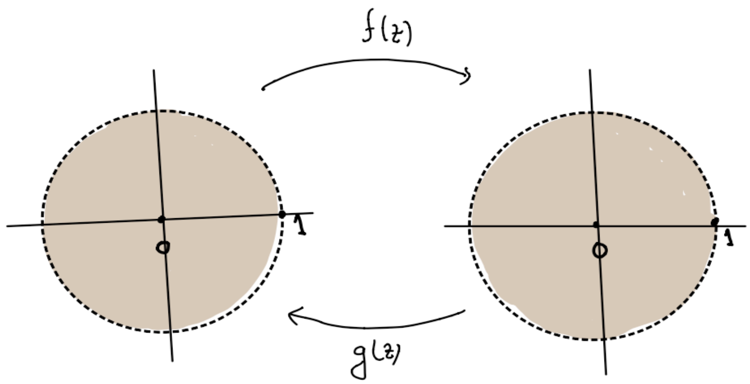


*Example 3* Same question for $A$ equal to the open unit disk, $B$ equal to the complement of the closed unit disk.

one pair: $f(z) = \dfrac{1}{z}$, $g(w) = \dfrac{1}{w}$



*Example 4* Filled in, because we did this earlier in the course and also partly in HW, back in October, so it's review! How many conformal bijections are there between the unit disk and itself?




*step 1:* Let $f$: $D(0; 1) \rightarrow D(0; 1)$ be a conformal bijection with $f(0) = 0$. Then $f(z) = e^{i\theta} z$ is a rotation about the origin by the angle $\theta$.
*proof:* Let $f$ be such a conformal bijection. Consider the auxillary functions

$$g(z) := \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

$$h(z) := \begin{cases} \dfrac{f^{-1}(z)}{z} & z \neq 0 \\ (f^{-1})'(0) = \dfrac{1}{f'(0)} & z = 0 \end{cases}$$

Then $g$ and $h$ are continuous at the origin so have removable singularities, i.e. are analytic. On any circle of radius $r < 1$

$$\left|g(z)\right| \leq \frac{1}{r}, \ |h(z)| \leq \frac{1}{r}$$

and so as $r$ increases to 1 we deduce from the maximum principle that on the entire $D(0; 1)$ we have upper bounds

$$|g(z)| \leq 1, \quad \text{i.e.} \quad |f(z)| \leq |z|$$

and

$$|h(w)| \leq |w| \quad \text{i.e.} \ |f^{-1}(w)| \leq |w| \Rightarrow |z| \leq |f(z)| \quad (\text{let } w = f(z))$$

Thus $|g(z)| \equiv 1$ so by part 2 of the Maximum Principle, $g(z) = e^{i\theta}$ and

$$f(z) = e^{i\theta} z.$$

*step 2:* For $z_0 \in \mathrm{D}(0;1)$, consider the *Mobius transformation*

a) $g_{z_0}(z) := \dfrac{z_0 + z}{1 + \bar{z}_0 z}$.

Then $g_{z_0}(z)$ is conformal in the closed unit disk with $g_{z_0}{}'(z) = \dfrac{1 - |z_0|^2}{(1 + \bar{z}_0 z)^2}$ existing and non zero in the closed unit disk.

Notice that $g(0) = z_0$. Also, we checked that $g$ transforms the unit circle to the unit circle, so that by the maximum modulus principle, $|g(z)| < 1 \ \forall\, z \in \mathrm{D}(0;1)$: For $|z| = 1$,

$$\left|\frac{z_0 + z}{1 + \bar{z}_0 z}\right|^2 = \frac{(z_0 + z)(\bar{z}_0 + \bar{z})}{(1 + \bar{z}_0 z)(1 + z_0 \bar{z})} = \frac{|z_0|^2 + z\,\bar{z}_0 + z_0 \bar{z} + |z|^2}{1 + \bar{z}_0 z + z_0 \bar{z} + |z_0|^2 |z|^2} = 1.$$

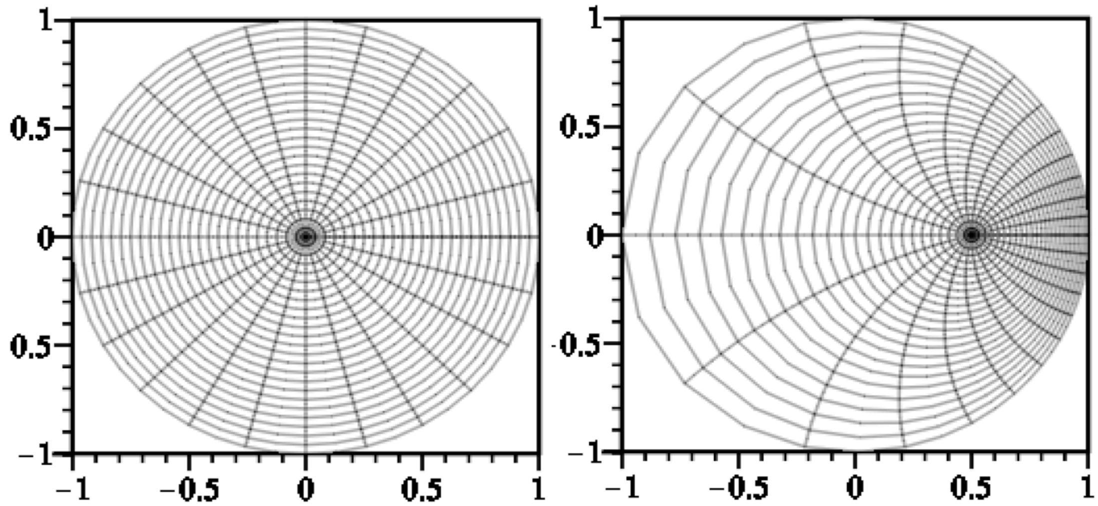
b) We solved for the inverse function and it turns out to be the Mobius transformation $g_{-z_0}(z)$:

$$\frac{z_0 + z}{1 + \bar{z}_0 z} = w \quad \text{iff } z_0 + z = w(1 + \bar{z}_0 z) \quad \text{iff } z = \frac{-z_0 + w}{1 - \bar{z}_0 w}$$

$$g_{-z_0}(w) = \frac{-z_0 + w}{1 - \bar{z}_0 w}.$$

Combining (a), (b) we see that the $g_{z_0}(z), g_{-z_0}(z)$ are inverse conformal diffeomorphisms of the unit disk.

The transformation picture for $g_{\frac{1}{2}}(z)$ and $g_{-\frac{1}{2}}(z)$. Which is which?




<u>Step 3</u>: Combining steps 1 and 2, we showed that for each $z_0 \in \mathrm{D}(0;1)$ *every* conformal diffeomorphism of the unit disk with

$$f(0) = z_0$$

can be written as

$$f(z) = g_{z_0}\left(e^{i\theta} z\right)$$

for some choice of θ. Proof: The composition function

$$g_{-z_0} \circ f$$

is a conformal diffeomorphism of the unit disk which maps the origin to itself. Thus

$$g_{-z_0} \circ f(z) = e^{i\theta} z$$

$$f(z) = g_{z_0}\left(e^{i\theta} z\right).$$

Riemann Mapping Theorem (version 1)
Let $A \subseteq \mathbb{C}$ (but $A \neq \mathbb{C}$) be open and simply connected.
Let $z_0 \in A$. Let $\theta \in (-\pi, \pi]$.
Then $\exists !\, f : A \to \mathrm{D}(0; 1)$ such that $f$ is a conformal bijection satisfying

$$f(z_0) = 0$$
$$arg(f'(z_0)) = \theta$$

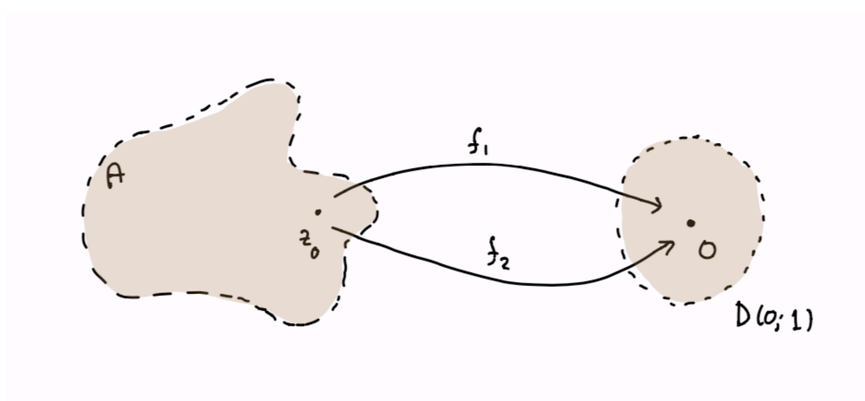
Note that this means there are three real degrees of freedom for conformal bijections with the disk: 2 from the choice of $z_0$ and one from the choice of the argument of $f'(z_0)$. So we probably missed some of the possibilities in our early examples. But we just proved both existence and uniqueness for conformal transformations of the unit disk though, once $z_0$ and $\theta$ are specified, using Mobius transformations and rotations.

The existence part of the general proof for any open simply connected subset of $\mathbb{C}$ except $\mathbb{C}$ itself would take several lectures to explain and we won't do it in this course. But we already have the tools to prove uniqueness:

*proof of uniqueness*: Suppose $f_1, f_2$ satisfy the conditions above. Define

$$g := f_2 \circ f_1^{-1} : \mathrm{D}(0; 1) \to \mathrm{D}(0; 1).$$

Use our earlier discussion.



Riemann Mapping Theorem (version 2)

Let $A, B \subseteq \mathbb{C}$ be open and simply connected but not all of $\mathbb{C}$.

Let $z_0 \in A$, $w_0 \in B$, $\theta \in (-\pi, \pi]$.

Then $\exists! f: A \to B$ such that $f$ is a conformal bijection satisfying

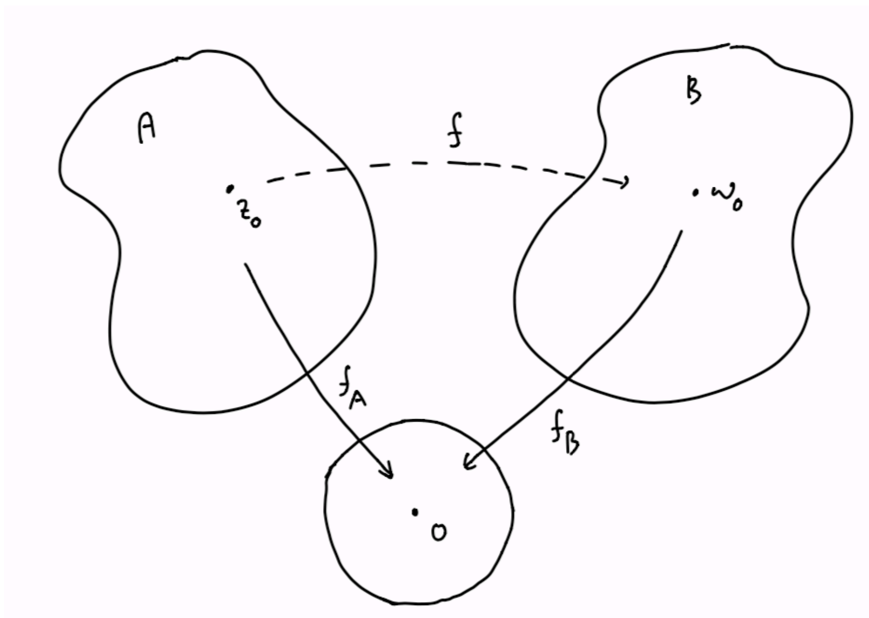
$$f(z_0) = w_0$$

$$arg(f'(z_0)) = \theta$$

*proof:* Chase the diagram arrows below to prove existence and uniqueness from version1 of the RMT: Letting $f_A, f_B$ be as in version 1 on the previous page, say with arguments of the derivative at $z_0$ both equal to zero; along with a rotation of the unit disk. In other words, show that

$$f(z) = f_B^{-1}\left(e^{i\theta} f_A(z)\right)$$

is the unique conformal transformation that works.



It turns out that the maps we were missing in the examples from the start of class were compositions of the ones we found, with *fractional linear transformations*, of which Mobius transformations are examples.

Def a *fractional linear transformation (FLT)* $f:\mathbb{C}\cap\{\infty\}\to\mathbb{C}\cup\{\infty\}$ is a meromorphic function defined by

$$f(z)=\frac{a\,z+b}{c\,z+d},$$

where $a$, $b$, $c$, $d\in\mathbb{C}$ and

$$ad-bc=det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\neq 0.$$

Note that when the determinant does equal zero, the function $f$ is just a constant. Also, one could normalize the determinant to be 1 by dividing all of the coefficients by the same number (a square root of the determinant).

Example $f(z)=a\,z+b=\dfrac{a\,z+b}{0\,z+1}$. You will show in your homework that these are the only one-to-one conformal maps defined on all of $\mathbb{C}$. Notice that they are conformal bijections of $\mathbb{C}$.

Exercise Why is there no conformal bijection $f:\mathbb{C}\to\mathrm{D}(0;\,1)$ ? It's a one-line answer if you can think of it.

The algebra of fractional linear transformations: Show that if

$$f(z) = \frac{a\,z + b}{c\,z + d}$$

$$g(w) = \frac{\alpha\,w + \beta}{\gamma\,w + \delta}$$

Then

$$g(f(z)) = \frac{A\,z + B}{C\,z + \mathrm{D}}$$

where

$$\begin{bmatrix} A & B \\ C & \mathrm{D} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

A matrix for the composition FLT can be obtained by multiplying the matrices for the individual FLTs!

Geometers would say: The *group* $SL(2, \mathbb{C})$ ("The *special linear group* of $2 \times 2$ matrices with entries from $\mathbb{C}$) *acts* on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The word *special* refers to the fact that each FLT can be represented uniquely with a matrix having determinant exactly equal to $1$, and the word *acts* refers to the fact that matrix multiplication (the group operation) in the group, corresponds to composition of transformations on the Riemann sphere.

To be continued .... this algebra makes composing and finding inverses for FLT's straightforward. On Wednesday we'll continue this discussion and explain interesting geometry related to FLT's.